P-odd Spectral Density at Weak Coupling

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Axial Charge is P- and CP-odd

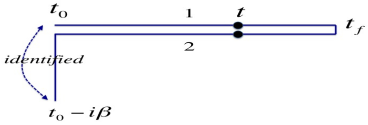
$$\begin{aligned} \psi_L &\rightarrow q_L\,(\mathrm{quark})\,,\ \bar{q}_R\,(\mathrm{anti-quark}) \\ \psi_R &\rightarrow q_R\,(\mathrm{quark})\,,\ \bar{q}_L\,(\mathrm{anti-quark}) \end{aligned}$$

$$\begin{aligned} P &: \quad q_L \longleftrightarrow q_R\,,\ \bar{q}_L \longleftrightarrow \bar{q}_R \\ C &: \quad q_L \longleftrightarrow \bar{q}_L\,,\ q_R \longleftrightarrow \bar{q}_R \end{aligned}$$

Axial Charge

$$J^0_A = N(q_L) + N(\bar{q}_L) - N(q_R) - N(\bar{q}_R)$$

Schwinger-Keldysh contour



$$G^{ij}_{12}(x) = \langle J^i_1(x) J^j_2(0)\rangle = \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}\hat{J}^j(0)\hat{J}^i(x))$$

$$G^{ij}_{21}(x) = \langle J^i_2(x) J^j_1(0)\rangle = \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}\hat{J}^i(x)\hat{J}^j(0))$$

$$G^{ij}_{11}(x) = \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}\mathcal{T}(\hat{J}^i(x)\hat{J}^j(0)))\,,\,\mathcal{T} = \mathrm{time\ ordering}$$

$$G^{ij}_{22}(x) = \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}\bar{\mathcal{T}}(\hat{J}^i(x)\hat{J}^j(0)))\,,\,\bar{\mathcal{T}} = \mathrm{anti-time\ ordering}$$

“ra”-Variables

Basic (Thermal) Relations

$$G^{ij}_{21}(\omega) = e^{\beta\omega} G^{ij}_{12}(\omega)$$
$$G_{11} + G_{22} = G_{12} + G_{21}$$

In terms of “ra”-variables

$$J_r = \frac{1}{2}(J_1 + J_2)\,, \quad J_a = J_1 - J_2$$

they become

$$G^{ij}_{rr}(\omega) = \left(\frac{1}{2} + n_B(\omega)\right)\left(G^{ij}_{ra} - G^{ij}_{ar}\right)$$
$$G^{ij}_{aa} = 0$$

Note: Retarded Function is $G^{ij}_R = (-i)G^{ij}_{ra}$

Current-Current Spectral Density

Since $G_R^{ij}(x)$ is real-valued, we have

$$G_R^{ij}(\omega,\vec{k}) = (G_R^{ij}(-\omega,-\vec{k}))^*.$$

This implies $G_{ar}^{ij}(\omega,\vec{k}) = G_{ra}^{ji}(-\omega,-\vec{k}) = -(G_{ra}^{ji}(\omega,\vec{k}))^*$

Define spectral density as

$$\rho^{ij}(\omega,\vec{k}) \equiv G_{ra}^{ij} - G_{ar}^{ij} = G_{ra}^{ij} + (G_{ra}^{ji})^*$$

- **It is a hermitian matrix in terms of i,j indices**
- **It is an odd function of ω**

Then we have

$$G_{rr}^{ij}(\omega,\vec{k}) = \left(\frac{1}{2} + n_B(\omega)\right)\rho^{ij}(\omega,\vec{k})$$

Governs Thermal Current Fluctuations

P-odd Spectral Density

$$\rho^{ij}(\omega,\vec{k}) = \rho^{ij}_{\mathrm{even}}(\omega,\vec{k}) + i\epsilon^{ijl}k^l\rho_{\mathrm{odd}}(\omega,\vec{k})$$

- **P-odd part is purely imaginary**
- ρ_{odd} **is an odd function of** ω
- **Since** ρ^{ij} **is C-even,** ρ_{odd} **is C-even**
- **Since** $\epsilon^{ijl}k^l$ **is P-odd, we see that** ρ_{odd} **is P and CP-odd, the same quantum number as the axial charge**

$$\rho_{\mathrm{odd}}(\omega,\vec{k}) \propto \mu_A$$

Relation to Chiral Magnetic Conductivity

Recall Chiral Magnetic Conductivity (Kharzeev-Warringa)

$$\vec{J}(\omega,\vec{k}) = \sigma_\chi(\omega,\vec{k})\vec{B}(\omega,\vec{k})$$

This implies $G_R^{ij} \sim i\epsilon^{ijl}k^l\sigma_\chi(\omega,\vec{k})$

Since $G_R = (-i)G_{ra}$**, we have**

$$\begin{aligned}\rho^{ij} &= G_{ra}^{ij} + (G_{ra}^{ji})^* \sim -\left(\sigma_\chi(\omega,\vec{k}) - (\sigma_\chi(\omega,\vec{k}))^*\right)\epsilon^{ijl}k^l \\ &= -2i\,\mathrm{Im}[\sigma_\chi(\omega,\vec{k})]\epsilon^{ijl}k^l\end{aligned}$$

$$\rho_{\rm odd}(\omega,\vec{k}) = -2\,\mathrm{Im}[\sigma_\chi(\omega,\vec{k})]$$

Note that chiral anomaly protected value $\lim_{\omega,\vec{k}\to 0}\sigma_\chi(\omega,\vec{k}) = \frac{\mu_A}{2\pi^2}$ **doesn't enter the spectral density**

Two Physical Examples

Second Order Transport from Chiral Anomaly

(Kharzeev-HUY, Jimenez-Alba-HUY)

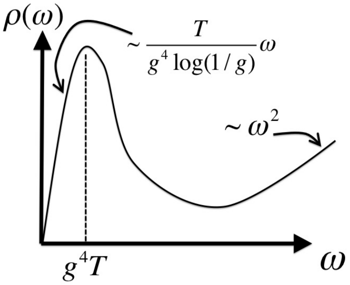
$$\vec{J} = \frac{\mu_A}{2\pi^2}\vec{B} + \xi_5 \frac{d\vec{B}}{dt} \longrightarrow \rho_{\rm odd}(\omega, \vec{k}=0) \sim 2\xi_5\,\omega + \cdots$$

P- and CP-odd Observables in Photon Emissions

(Mamo-HUY)

$$A_{\pm\gamma} \equiv \frac{\frac{d\Gamma}{d^3\vec{k}}(\epsilon_+) - \frac{d\Gamma}{d^3\vec{k}}(\epsilon_-)}{\frac{d\Gamma}{d^3\vec{k}}(\epsilon_+) + \frac{d\Gamma}{d^3\vec{k}}(\epsilon_-)} = \frac{-\omega\,\rho_{\rm odd}(\omega, |\vec{k}|=\omega)}{\mathrm{Im}\mathrm{Tr}G_R}$$

Leading Order Weak Coupling Computations



The coefficient ξ_5 is non-analytic in coupling constant

$$\xi_5 \sim \frac{1}{g^4 \log(1/g)}$$

which is similar to electric conductivity $\sigma \sim \frac{e^2 T}{g^4 \log(1/g)}$

In free theory, $\mathrm{Im}\,\sigma_\chi(\omega, \vec{k} = 0) \sim \frac{1}{3\pi}\delta(\omega)\mu_A\omega$

(Kharzeev-Warringa) , so $\xi_5^{\mathrm{free}} = -\frac{1}{3\pi}\delta(\omega = 0)\mu_A$

Brief Sketch of Key Physics

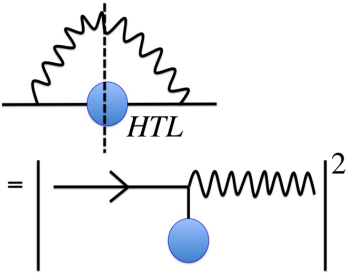
Charge transport is essentially proportional to the mean free path $l_{\rm mfp}$ of hard ($p \sim T$) particles, which is governed by inverse scattering rate

$$l_{\rm mfp} \sim \frac{1}{\sigma_{\rm eff}}$$

The total cross section (damping rate) $\sigma_{\rm total} \sim g^2 \log(1/g)$ is dominated by thermally excited magnetic gauge field with ultra-soft momentum kicks $q \ll gT$. However, this implies small deflection of charge carrier motion $\theta \sim \frac{q}{p} \ll g$, which should not affect charge transport. One effectively has

$$\sigma_{\rm eff} \sim \int d^3q \frac{d\Gamma}{d^3q}\, \theta^2 \sim \int d^3q \frac{d\Gamma}{d^3q} \left(\frac{q}{p}\right)^2 \sim g^4 \log(1/g) T$$

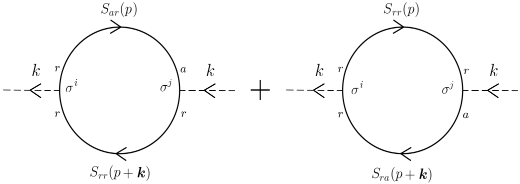
Another important process is the charge carrier fermions converting to gauge field by scattering with soft thermal anti-fermions



This rate also turns out to be $g^4 \log(1/g)T$

In any case, the mean free path $l_{\rm mfp} \sim \frac{1}{g^4 \log(1/g)T}$ **is non-analytically long**

Naturally, the singular contributions from these long-lived charge carriers arise from nearly on-shell propagators in the diagrams.



1-loop real-time diagrams contain "Pinch Singularity"

$$\int dp^0 \; S_{ra}(p) S_{ar}(p) \sim \int dp^0 \; \frac{1}{p^0 - |\vec{p}| + i\epsilon} \frac{1}{p^0 - |\vec{p}| - i\epsilon} \sim \frac{1}{\epsilon}$$

This is regulated by the damping rate

$\epsilon \to \zeta/2 \sim g^2 \log(1/g)$ (**Jeon**)

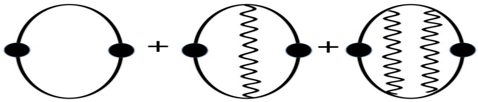
1-Loop Result for ξ_5

$$\xi_5^{1-\mathrm{loop}} = -\frac{\mu}{6\pi^2\zeta} \sim \frac{1}{g^2\log(1/g)}$$

How do we obtain $\sim \frac{1}{g^4\log(1/g)}$ from this ???

Ladder Summation

There are other diagrams with the same $1/g^2$ dependence (Jeon, Basagoiti)



This gives arise to a geometric summation

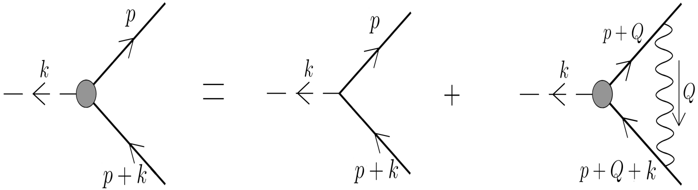
$$\xi_5 \sim \frac{1}{\zeta} + \frac{1}{\zeta}\left(g^2\mathcal{K}\frac{1}{\zeta}\right) + \frac{1}{\zeta}\left(g^2\mathcal{K}\frac{1}{\zeta}\right)^2 + \cdots = \frac{1}{\zeta}\frac{1}{1-g^2\mathcal{K}/\zeta}$$

Since $g^2\mathcal{K}/\zeta \sim \mathcal{O}(g^0)$ we would still expect $1/g^2$.

However, as it turns out, $g^2\mathcal{K}/\zeta = 1$ at leading order in g, which has its origin in Ward Identity (Aarts-Resco)

We need the next leading order $g^2\mathcal{K}/\zeta = 1 - \# g^2$ and $\xi_5 \sim \frac{1}{\zeta}\frac{1}{\# g^2} \sim \frac{1}{g^4 \log(1/g)}$

Schwinger-Dyson Equation



$$D(p,k) = 1 + g^2 \int \frac{d^4Q}{(2\pi)^4} \, \frac{\mathcal{K}(p,Q,k)}{\zeta} \, D(p+Q,k)$$

$$\zeta^{\rm sf}_{\vec{p},s}\chi_s(|\vec{p}|) =$$

$$s - \frac{e^2 m_D^2 \log(1/e)}{4\pi}\left(\frac{1}{\beta|\vec{p}|^2}\chi_s(|\vec{p}|) - \left(\frac{1}{\beta|\vec{p}|} + n_s(|\vec{p}|) - \frac{1}{2}\right)\chi_s'(|\vec{p}|) - \frac{1}{2\beta}\chi_s''(|\vec{p}|)\right)$$

QCD with $N_F = 2$ flavors

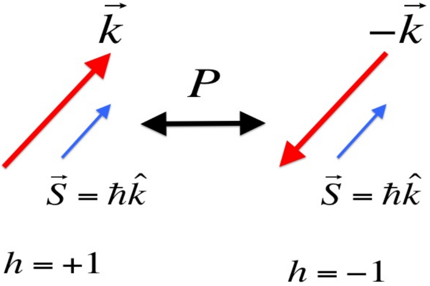
$$\xi_5^{\rm QCD} = -\frac{2.003}{g^4 \log(1/g)} \frac{\mu_A}{T}$$

We always seem to get a relative negative sign between σ_0 and ξ_5

$$\vec{J} = \sigma_0 \vec{B} + \xi_5 \frac{d\vec{B}}{dt}$$

which means that chiral magnetic current is resistant to the change of magnetic field

P- and CP-odd Photon Observable at Weak Coupling (Mamo-HUY)



Spin polarization (helicity h) is P-odd

$$\epsilon^{\mu}_{\pm} = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)\,, \quad \vec{k} = k\hat{x}^3$$

P- and CP-odd Observable

$$A_{\pm\gamma} \equiv \frac{\frac{d\Gamma}{d^3\vec{k}}(\epsilon_+) - \frac{d\Gamma}{d^3\vec{k}}(\epsilon_-)}{\frac{d\Gamma}{d^3\vec{k}}(\epsilon_+) + \frac{d\Gamma}{d^3\vec{k}}(\epsilon_-)}$$

Using

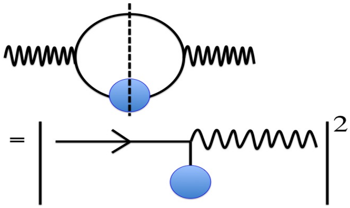
$$\frac{d\Gamma}{d^3\vec{k}}(\epsilon^\mu) = \frac{e^2}{(2\pi)^3 2|\vec{k}|}\frac{-2}{e^{\beta|\vec{k}|}-1}\mathrm{Im}\left[(\epsilon^\mu)^*\epsilon^\nu G^R_{\mu\nu}\right]_{k^0=|\vec{k}|}$$

we have

$$A_{\pm\gamma} = \frac{\mathrm{Im}G^R_+ - \mathrm{Im}G^R_-}{\mathrm{Im}G^R_+ + \mathrm{Im}G^R_-} = \frac{2\,\mathrm{Re}G^R_{12}}{\mathrm{Im}\mathrm{Tr}G^R}$$

where $G^R_\pm \equiv G^R_{11} \pm iG^R_{12}$

As is the case for the total emission rate, the leading log rate comes from the fermion conversion to hard photon



$$\frac{d\Gamma(\epsilon^+)}{d^3k} - \frac{d\Gamma(\epsilon^-)}{d^3k} = \frac{e^2}{(2\pi)^4 2\omega} n_B(\omega)(n_+(0) - n_-(0)) m_f^2 \log(1/g)$$

where $m_f^2 = \frac{g^2T^2}{4}\frac{N_c^2-1}{2N_c}$ **is the thermal fermion mass. These logs will match with those coming from hard Compton and Pair-annihilation diagrams, giving rise to** $\log\left(\frac{E}{g^2T}\right)$ **form**

Physics of The Result ?

The total leading log emission rate is given by

$$\frac{d\Gamma(\epsilon^+)}{d^3k}+\frac{d\Gamma(\epsilon^-)}{d^3k}=\frac{e^2}{(2\pi)^4 2\omega}n_B(\omega)(n_+(0)+n_-(0))m_f^2\log(1/g)$$

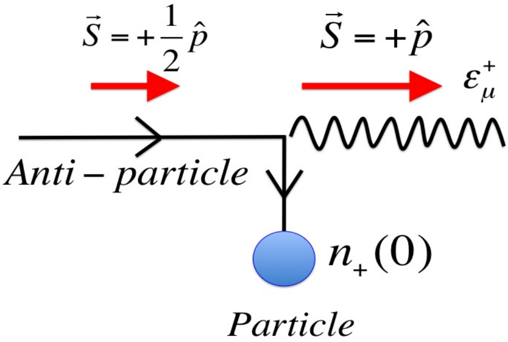
while the difference is

$$\frac{d\Gamma(\epsilon^+)}{d^3k}-\frac{d\Gamma(\epsilon^-)}{d^3k}=\frac{e^2}{(2\pi)^4 2\omega}n_B(\omega)(n_+(0)-n_-(0))m_f^2\log(1/g)$$

This means that

$$\begin{aligned}\frac{d\Gamma(\epsilon^+)}{d^3k} &= \frac{e^2}{(2\pi)^4 2\omega}n_B(\omega)n_+(0)m_f^2\log(1/g)\\ \frac{d\Gamma(\epsilon^-)}{d^3k} &= \frac{e^2}{(2\pi)^4 2\omega}n_B(\omega)n_-(0)m_f^2\log(1/g)\end{aligned}$$

Angular Momentum Conservation



To conserve angular momentum, the incoming fermion must be an anti-particle with $\vec{S} = +\frac{1}{2}\hat{p}$. Then, the annihilating soft fermion should be a particle whose distribution is given by $n_+(0)$ at soft momentum

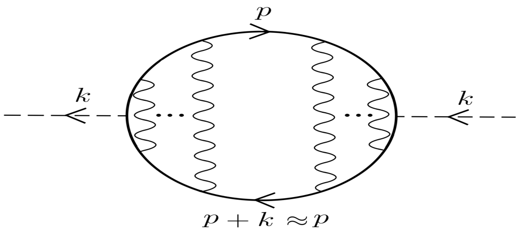
Complete Leading Order at Weak Coupling

For complete leading order $\frac{d\Gamma}{d^3k} \sim e^2 g^2$, we need

- **Hard Compton and Pair Annihilation contributions**
- **Collinear Bremstrahlung and Annihilation (LPM) contributions**

As it turns out, the latter is easier than the first as the modification for this from the previous total rate computation is very simple

Collinear P-odd Bremstrahlung

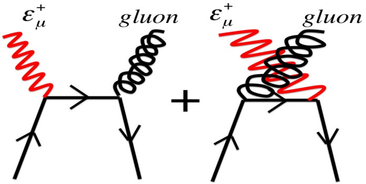


We get the same Schwinger-Dyson equation as in the literature (Arnold-Moore-Yaffe)

$$
\begin{aligned}
-i\delta E_{p_\perp} f^\mu(p,k) &= \frac{p^\mu}{p_\parallel} + g^2\frac{N_c^2-1}{2N_c}\int\frac{d^4Q}{(2\pi)^4}\delta(q^0-q_\parallel)\frac{p^\alpha p^\beta}{p_\parallel^2}\\
&\times\ G^{rr}_{\alpha\beta}(Q)\left(f^\mu(p_\parallel,p_\perp+q_\perp,k)-f^\mu(p_\parallel,p_\perp,k)\right)
\end{aligned}
$$

the only change is the vertex and distribution functions in the final rate computation

Compton and Pair Annihilation



For Pair Annihilation diagram, we have

$$\frac{d\Gamma(\epsilon^+)}{d^3k} - \frac{d\Gamma(\epsilon^-)}{d^3k} = \frac{(N_c^2-1)/2}{(2\pi)^3 2\omega}\int_p\int_{p'}\int_{k'}(|\mathcal{M}^+|^2 - |\mathcal{M}^-|^2)$$
$$\times \quad n_+(p)n_-(p')(1+n_B(k'))(2\pi)^4\delta(p+p'-k-k')$$

where

$$|\mathcal{M}^+|^2 - |\mathcal{M}^-|^2 = 4(t-u)\left((1/t+1/u) - 2\left(p_\perp/t - p'_\perp/u\right)^2\right)$$

We need numerical integration
(Kiminad Mamo is looking into it)

Future Direction?

- Chiral Kinetic Theory Computation
- Lattice Computation
- Other Observables at Weak Coupling ?

Thank You for Listening